Research ideas from effective resistance

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NYCU Colloquium
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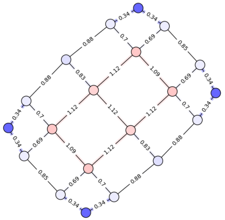
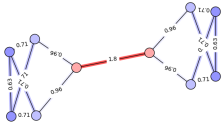




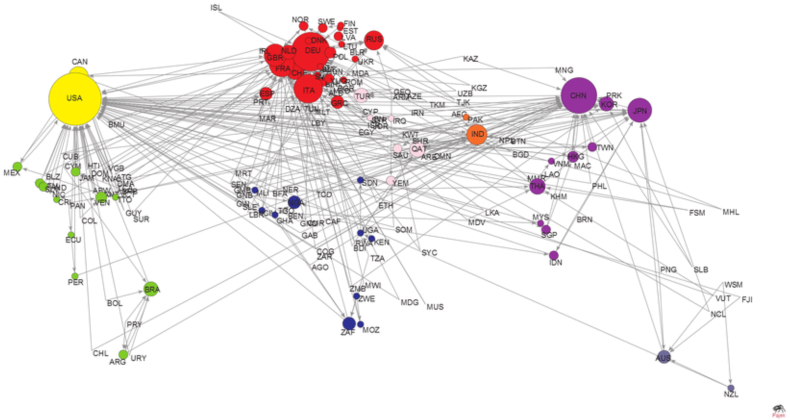
Background

Past 12 years:

- NYCU
- Taipei, National Center for Theoretical Sciences
- Seattle, Fred Hutch Cancer Center
- Seattle, University of Washington
- Ann Arbor, University of Michigan

Motivation

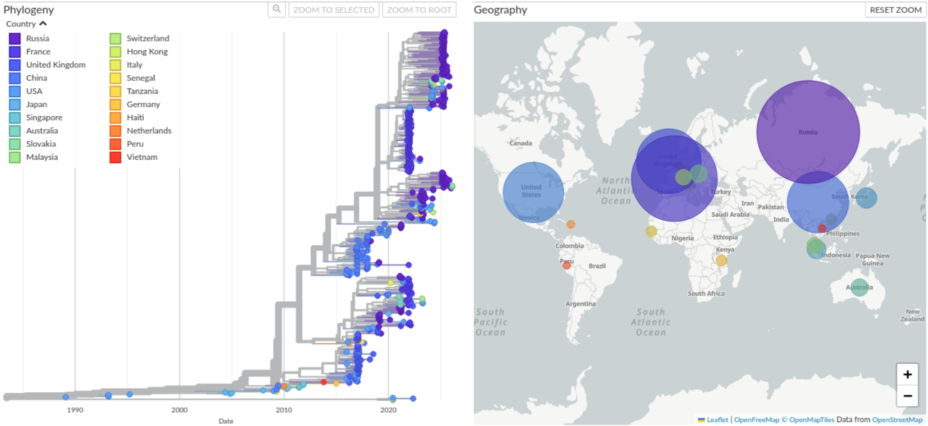
Problem: How to understand "geometry" of a graph?



(image from Silvia Nenci et al. 2014)

Motivation

Problem: How to understand virus evolution?



(image from Nextstrain, T. Bedford, R. Neher et al. 2026)

Motivation

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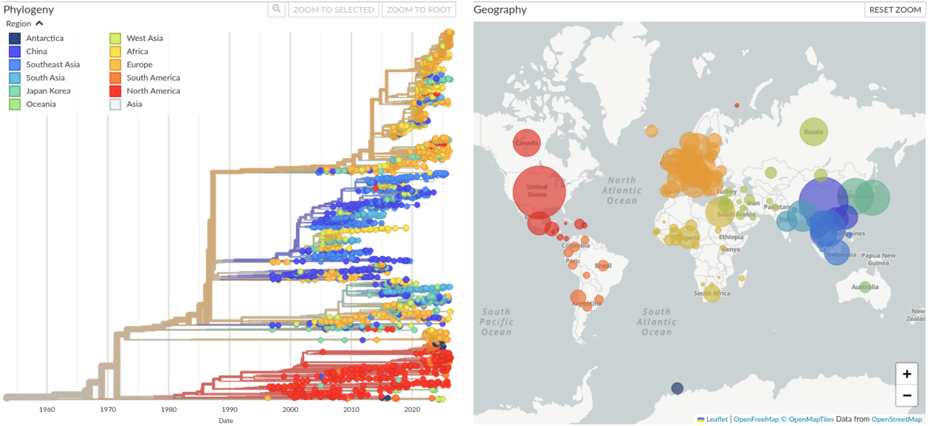

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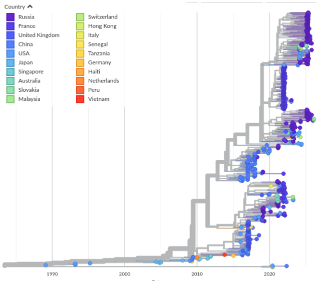
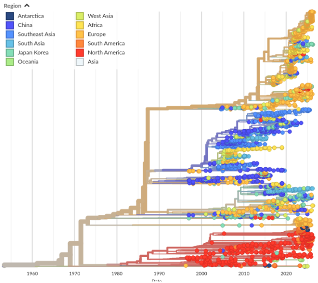
Problem: How to understand virus evolution?



$\leftarrow ? \rightarrow$



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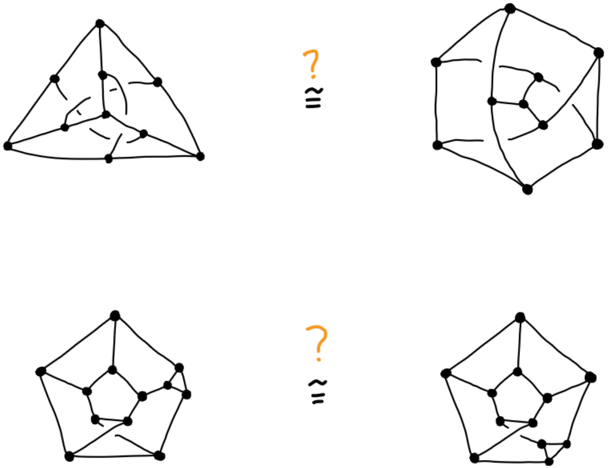
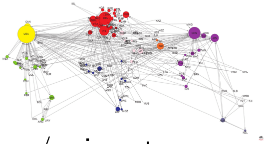
Motivation

Problem: How to find isomorphism between two graphs?



Motivation: graph curvature

Problem: How to understand “geometry” of a graph?



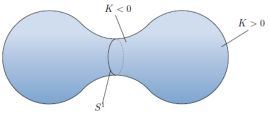
- Applied world: max flow / min cut, community detection

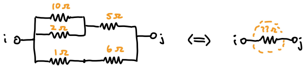
Differential geometry

curvature, Ricci flow

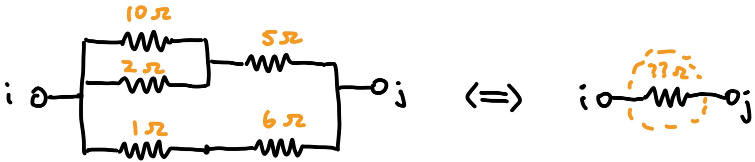
Combinatorics

effective resistance





Why effective resistance?



Close connections to:

- simple random walk on G
- uniformly random spanning trees on G

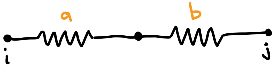
Recent breakthrough applications:

- graph sparsification (Spielman–Srivastava, 2009)
- traveling salesman problem (Anari–Oveis-Gharan, 2015)

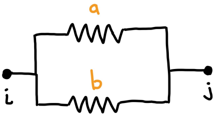
Effective resistance

Setting: graph $G = (V, E)$, each edge e has a positive resistance ℓ_e

How to compute the **effective resistance** ω_{ij} for vertices $i, j \in V$?

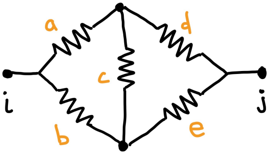
- series rule: 

$\omega_{ij} = a + b$

- parallel rule: 

$$\omega_{ij} = \frac{ab}{a + b}$$

- general case: Wheatstone bridge



$$\omega_{ij} = \frac{abd+abe+ade+bde+abc+ace+bcd+cde}{ad+ae+bd+be+ac+bc+cd+ce}$$

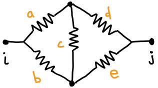
Effective resistance

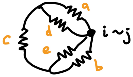
⚠ Series and parallel rules not sufficient to find effective resistance

- General case: use weighted sums of spanning trees

Theorem (Kirchhoff)

$$\omega_{ij} = \frac{\sum_{T(G/ij)} \prod_{e \notin T} \ell_e}{\sum_{T(G)} \prod_{e \notin T} \ell_e}$$

Example: $G =$ 

, $G/ij =$ 

$$\omega_{ij} = \frac{abd + abe + ade + bde + abc + ace + bcd + cde}{ad + ae + bd + be + ac + bc + cd + ce}$$

Effective resistance

Theorem (Rayleigh's law)

For any edge e and vertices i, j we have

$$\frac{\partial}{\partial \ell_e} \omega_{ij} \geq 0.$$

- physically "obvious"
- mathematically ...

$$\frac{\partial}{\partial c} \omega_e = \frac{\partial}{\partial c} \left(\frac{abd + abe + ade + bde + abc + ace + bcd + cde}{ad + ae + bd + be + ac + bc + cd + ce} \right) =?$$

Effective resistance

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$$= \left(\frac{ae - bd}{ad + ae + bd + be + ac + bc + cd + ce} \right)^2 \quad (!?)$$

Effective resistance

Theorem (Rayleigh's law)

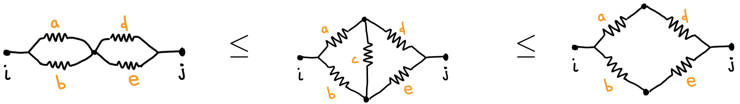
For any edge e and vertices i, j we have

$$\frac{\partial}{\partial \ell_e} \omega_{ij} \geq 0.$$

- delete edge $\leftrightarrow$ $\ell_e = +\infty$
- contract edge $\leftrightarrow$ $\ell_e = 0$

Corollary (usual Rayleigh's law)

$$\omega_{ij}(G/e) \leq \omega_{ij}(G) \leq \omega_{ij}(G \setminus e)$$



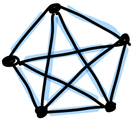
Curvature on graphs

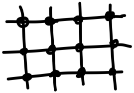
Definition

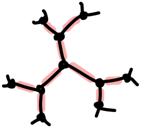
On weighted graph (G, ℓ), the *Foster–Ricci curvature* on edge e is

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

- Constant-curvature graphs: positive, zero, and negative







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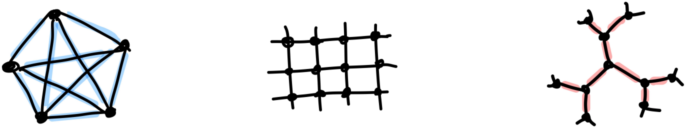

- Edge curvature gives more information than node curvature:

Curvature on graphs

Definition

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$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

Consider resulting Ricci flow

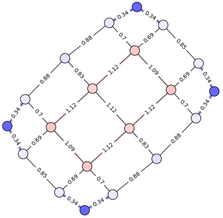
$$\frac{d}{dt}\ell_e(t) = -\mathrm{K}_e(t)$$

where $\mathrm{K}_e(t) = \mathrm{K}_e(\ell(t))$.

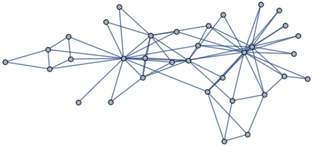
What does Ricci flow look like?

What does Ricci flow look like?

Ex. Finite grid



Ex. "Karate club" graph



(image from Wolfram Data Repository)

Properties of Ricci flow

	manifolds	graphs
Invariant under scaling	yes	yes
Volume decreasing when curvature nonnegative	yes*	yes (any graph)
Nonnegative curvature preserved under flow	yes	yes! (2025)
No periodic orbits	yes	Open

Ricci flow: open questions

Definition

The *Foster–Ricci flow* on weighted graph (G, ℓ) is

$$\frac{d}{dt}\ell_e(t) = -\mathrm{K}_e(t)$$

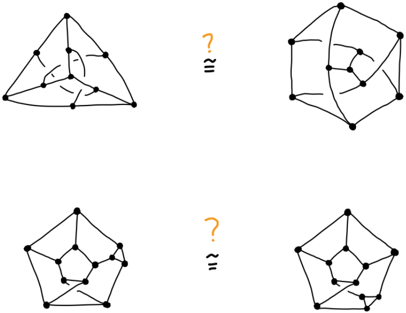
where edge curvature $$\mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

More questions:

- Are there any periodic orbits?
- Can we classify Einstein graphs? i.e. $\{\mathrm{K}_e : e \in E\} \sim \{\ell_e : e \in E\}$
- Cartesian products of graphs ... ?

Motivation: Weierstrass weights

Problem: How to find isomorphism between two graphs?

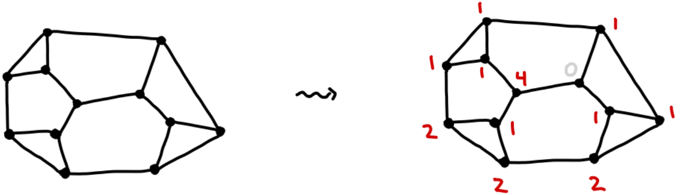


Weierstrass weights

- For finite graph $G = (V, E)$, Weierstrass weights give vertex labels

$$\mu : V \to \mathbb{Z}_{\geq 0}$$

<u>Ex.</u>



(genus = 7)

- Labels are bounded by genus := # independent cycles (a.k.a. cyclomatic #)

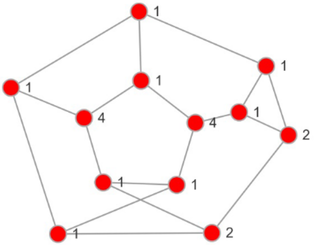
= # edges removed to get spanning tree $= |E| - |V| + 1$

Weierstrass weights

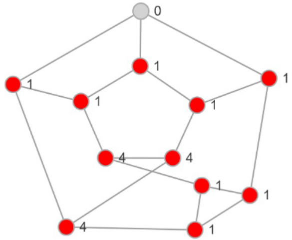
- For finite graph $G = (V, E)$, Weierstrass weights give vertex labels

$$\mu : V \to \mathbb{Z}_{\geq 0}$$

<u>Ex</u>.



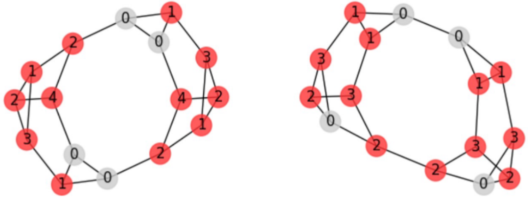
$$\not\cong$$



Weierstrass weights

- weights $\mu : V(G) \to \mathbb{Z}_{\geq 0}$ are isomorphism - invariant
- <u>Claim</u> : (informal) weights reflect mix of local + global structure

<u>Ex</u>. "Whitney twist" of graph two - sum



Weierstrass weights

How to compute Weierstrass weights?

- For chosen vertex v,
  - Compute reduced divisor at v via Dhar's algorithm
  - weight

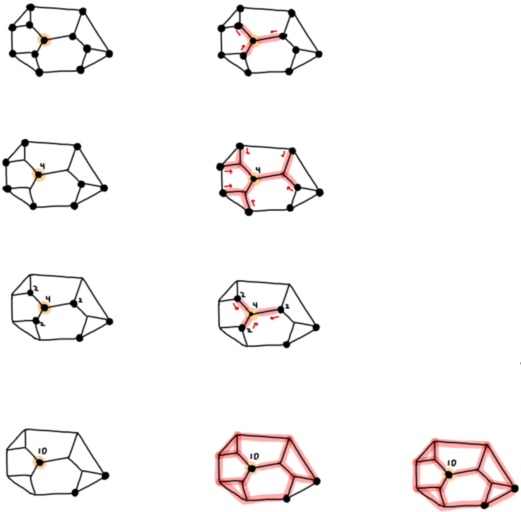
$$\mu(v) = (v\text{-coeff.}) - (\underbrace{g}_{\text{genus}} - 1)$$

Ex.

genus = 7, so $\mu(v) = 10 - 6 = 4$

Weierstrass weights

Ex.

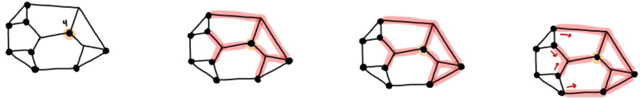


$$\mu(v) = 10 - (g-1)$$
$$= 4$$

→ DONE

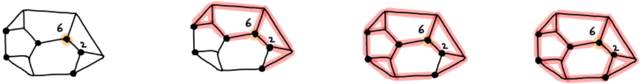
Weierstrass weights

Ex.



$\mu(v) = 6 - (g-1)$

$= 0$



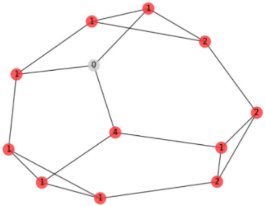
→ DONE

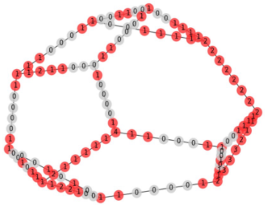
Weierstrass weights

Prop For any vertex, Weierstrass weight $\mu(v) \leq g-1$

↳ genus

Fact. Weierstrass weights are unchanged by "uniform refinement" of edges



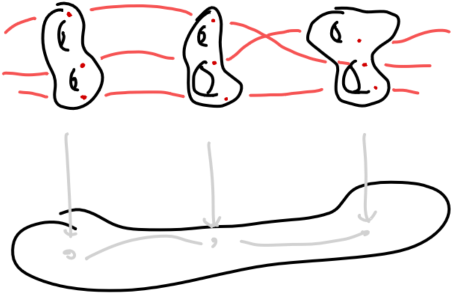


Fact Weierstrass weights are well-defined on associated metric graph, i.e. continuous space w/ edge $\simeq$ unit interval

Weierstrass weights

Goal: understand moduli space of algebraic curves $\mathcal{M}_g$

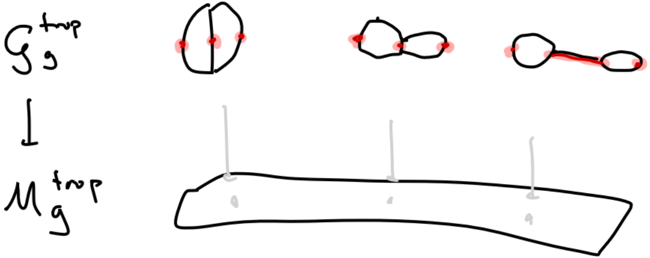
$$\begin{array}{c} \mathcal{C}_g \\ \downarrow \\ \mathcal{M}_g \end{array}$$



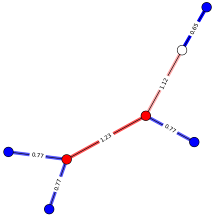
Slogan: "Weierstrass pts are coordinates on $\mathcal{M}_g$"

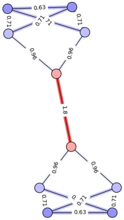
Weierstrass weights

Problem. What is "good" notion of tropical weight?

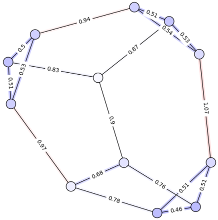


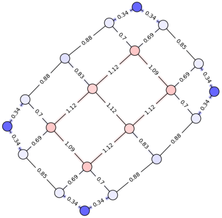
⚠ **Fact.** Tropical Weierstrass locus can be infinite, containing positive-length segments



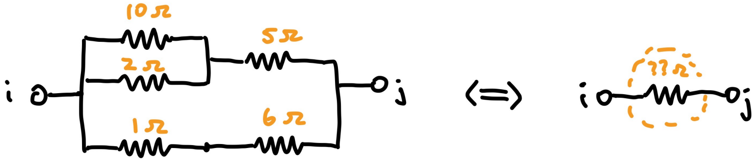


Thank you!





Effective resistance: quiz answer



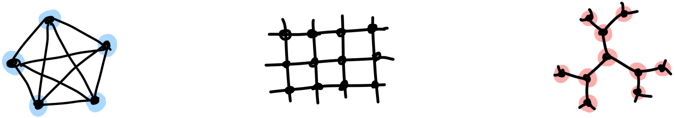
Answer: $\omega_{ij} = \dfrac{140}{41}$

Graph curvature: previous work

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

- A finite, vertex-transitive graph has (constant) positive node curvature.
- An infinite regular lattice is flat (zero curvature).
- An infinite tree has negative node curvature everywhere.



Graph curvature: previous work

(Devriendt–Lambiotte 2022) define **node curvature** at $i \in V$ as

$$p_i = 1 - \frac{1}{2} \sum_{e \ni i} \frac{\omega_e}{\ell_e}.$$

Can we make edge curvature "more local", in the sense that

$$p_i = \sum_{e \ni i} \mathrm{K}_{\vec{e}} \qquad \text{for edge curvatures } \mathrm{K}_{\vec{e}}?$$

Yes! Define

$$\text{oriented edge curvature} \qquad \mathrm{K}_{\vec{e}} = \frac{1}{\deg_i} - \frac{1}{2} \frac{\omega_e}{\ell_e}$$

$$\text{edge curvature} \qquad \mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

Why this graph curvature? Arithmetic geometry

Recall edge curvature $$\mathrm{K}_e = \frac{1}{\deg_i} + \frac{1}{\deg_j} - \frac{\omega_e}{\ell_e}$$

Problem: How to bound number of rational points on a curve?

Strategy:

- Embed curve X in Jacobian $Jac(X)$
- Take p-adic completion $\mathbb{Z} \to \mathbb{Z}_p$
- Consider special fiber (reduce mod p) $\mathbb{Z}_p \to \mathbb{F}_p$

Upshot:

- geometry of Jacobian $\rightsquigarrow$ geometry of curve
- local geometry over $\mathbb{Z}_p$ and $\mathbb{F}_p$ $\rightsquigarrow$ global geometry over $\mathbb{Z}$ and $\mathbb{Q}$

Why this graph curvature? Arithmetic geometry

Under tropicalization, the **Bergman measure** on alg. curve X becomes the **Zhang canonical measure** on the metric graph $\mathrm{trop}(X)$

- Bergman measure =
  pullback of flat metric on $Jac(X)$, under Abel–Jacobi
- Zhang canonical measure =

$$\mu_{\mathrm{can}} = \sum_{vinV} \left(1 - \frac{1}{2}\,\mathrm{deg}_v\right) \delta_v + \sum_{e \in E} \left(1 - \frac{\omega_e}{\ell_e}\right) \frac{dx}{\ell_e}$$

Upshot:

- curvature $\mathrm{K}_e = \dfrac{1}{\mathrm{deg}_i} + \dfrac{1}{\mathrm{deg}_j} - \dfrac{\omega_e}{\ell_e}$ is **discretization** of Zhang canonical measure

Why this graph curvature? Arithmetic geometry

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Theorem (Neeman, 1984)

On a smooth, projective algebraic curve, the generalized Weierstrass points of a high-degree divisor asymptotically distribute according to the **Bergman measure**

Why this graph curvature? Arithmetic geometry

Under tropicalization, the **Bergman measure** on alg. curve X becomes the **Zhang canonical measure** on the metric graph $\mathrm{trop}(X)$

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Theorem (R, 2024)

On a compact metric graph, the generalized Weierstrass points of a generic* high-degree divisor asymptotically distribute according to the **Zhang canonical measure**

Why this graph curvature? Arithmetic geometry

Other applications: **tau constant** is the "self-interaction potential" of the Zhang canonical measure

$$\tau(\Gamma) = \int\int_{\Gamma\times\Gamma} r(x,y)\, d\mu_{\mathrm{can}}(x) d\mu_{\mathrm{can}}(y)$$

Theorem (Cinkir, 2010)

The effective Bogomolov conjecture holds over function fields of characteristic zero.

Proof sketch. Prove certain lower bounds on $\tau(\Gamma)$

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The effective Bogomolov conjecture holds over function fields of characteristic zero.

Proof sketch. Prove certain lower bounds on $\tau(\Gamma)$

Conjecture (Rumely, 2000s)

For any compact metric graph Γ, we have the lower bound

$$\tau(\Gamma) \geq \frac{1}{108}\ell(\Gamma).$$